

Subvarieties in non-compact hyperkähler manifolds

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Abstract

Let M be a hyperkähler manifold, not necessarily compact, and $S \cong \mathbb{C}P^1$ the set of complex structures induced by the quaternionic action. Trianalytic subvariety of M is a subvariety which is complex analytic with respect to all $I \in S$. We show that for all $I \in S$ outside of a countable set, all compact complex subvarieties $Z \subset (M, I)$ are trianalytic. For M compact, this result was proven in [V1] using Hodge theory.

1 Introduction

Hyperkähler manifold is a Riemannian manifold with an algebra \mathbb{H} of quaternions acting in its tangent bundle, in such a way that for any almost complex structure induced by a quaternion $L \in \mathbb{H}$, $L^2 = -1$, the corresponding manifold (M, L) is Kähler. Such quaternions naturally form a 2-dimensional sphere $S^2 \cong \mathbb{C}P^1$,

$$\{L = aI + bJ + cK \mid a^2 + b^2 + c^2 = 1\}.$$

In the same way as the usual complex structure induces a $U(1)$ -action in the cohomology $H^*(M)$ of a compact Kähler manifold, the quaternionic structure induces an action of $SU(2) \subset \mathbb{H}$ on $H^*(M)$. This action is a hyperkähler analogue of the Hodge decomposition of the cohomology.

Hyperkähler manifolds were discovered by E. Calabi ([Ca]).

Using the Hodge theory and the $SU(2)$ -action on cohomology, many properties of hyperkähler manifolds were discovered ([V1], [V2] [V3]).

When an algebraic manifold is non-compact, one has a mixed Hodge structure in its cohomology, playing the same role as the usual Hodge structure. A hyperkähler analogue of mixed Hodge structure is unknown. This is why the methods of Hodge theory are mostly useless when one studies non-compact hyperkähler manifolds. Overall, not much is known about topology

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and algebraic geometry of non-compact (e.g. complete) hyperkähler manifolds.

There is an array of beautiful works on volume growth and asymptotics of harmonic functions on complete Ricci-flat manifolds (see e.g. [Yau], [Sor], [CM]). The full impact of these results on algebraic geometry is yet to be discerned.

In applications, one is especially interested in the hyperkähler manifolds equipped with so-called **uniholomorphic** $U(1)$ -**action**, discovered by N. Hitchin ([H], [HKLR]).

Example 1.1: Let M be a hyperkähler manifold, equipped with $U(1)$ -action ρ . Assume that ρ acts by holomorphic isometries on (M, I) for a fixed complex structure I , and the pullback of any induced complex structure under $\rho(\lambda)$ satisfies

$$\rho(\lambda)^*(L) = \rho_{\mathbb{C}P^1}(\lambda, L),$$

where $\rho_{\mathbb{C}P^1}$ is a standard rotation of the set $S^2 \cong \mathbb{C}P^1$ of induced complex structures fixing I and $-I$. Then the $U(1)$ -action is called **uniholomorphic**.

Geometry of such manifolds is drastically different from that of compact hyperkähler manifolds. For most examples of M equipped with uniholomorphic $U(1)$ -action, the manifolds (M, L) are isomorphic, as complex manifolds, for all $L \neq \pm I$. In many cases, (M, L) are also algebraic, for all $L \in \mathbb{C}P^1$.

On the other hand, when M is compact, (M, L) is algebraic for a dense and countable set of $L \in \mathbb{C}P^1$, and non-algebraic for all induced complex structures outside of this set ([F]). In the compact case, (M, L) are pairwise non-isomorphic for most $L \in \mathbb{C}P^1$.

The following result is known for compact hyperkähler manifolds, but its proof is a fairly complicated application of Hodge theory and $SU(2)$ -action on cohomology. Given the difference in geometry of compact and non-compact hyperkähler manifolds, its generalization to the non-compact case is quite unexpected.

Recall that a subset $Z \subset M$ is called **trianalytic** if it is complex analytic in (M, L) for all $L \in \mathbb{C}P^1$. A trianalytic subvariety is hyperkähler whenever it is smooth.

Theorem 1.2: Let M be a hyperkähler manifold, not necessarily compact, and $S \cong \mathbb{C}P^1$ the set of induced complex structures. Then there

exists a countable set $S_0 \subset S$, such that for all compact complex subvarieties $Z \subset (M, L)$, $L \notin S_0$, the subset $Z \subset M$ is trianalytic.

Proof: See Claim 3.3 and Theorem 3.4. ■

Remark 1.3: We should think of $L \in S \setminus S_0$ as of generic induced complex structures. Then Theorem 1.2 states that all compact complex subvarieties of (M, L) are trianalytic, for generic $L \subset \mathbb{C}P^1$.

Remark 1.4: If L is a generic induced complex structure, (M, L) has no compact subvarieties except trianalytic subvarieties. Since trianalytic subvarieties are hyperkähler in their smooth points, their complex codimension is even. Therefore, such (M, L) has no compact 1-dimensional subvarieties. This implies that (M, L) is non-algebraic, or (M, L) has no compact subvarieties of positive dimension.

Remark 1.5: For many examples of hyperkähler manifolds equipped with uniholomorphic $U(1)$ -action (Example 1.1), (M, L) are algebraic, for all L . Also, in these examples, (M, L) are pairwise isomorphic for all $L \neq \pm I$. In this case, Theorem 1.2 clearly implies that all (M, L) , $L \neq \pm I$ have no compact subvarieties.

This also follows from [HKLR], where it was shown that a moment map μ for $U(1)$ -action on (M, I) gives a Kähler potential on (M, J) , for any $J \circ I = -I \circ J$. Then, μ is strictly plurisubharmonic on (M, J) , and by maximum principle, the manifold (M, J) has no compact subvarieties.

2 Trisymplectic area function and trianalytic subvarieties

Let M be a Kähler manifold, and g its Riemannian form. For each induced complex structure $L \in \mathbb{H}$, consider the corresponding Kähler form $\omega_L := g(\cdot, L\cdot)$. Let $Z \subset M$ be a compact real analytic subvariety, $\dim_{\mathbb{R}} Z = 2n$, such that

$$\lim_{\varepsilon \rightarrow 0} \text{Area } \partial Z_\varepsilon = 0, \quad (2.1)$$

where $Z_\varepsilon \subset Z$ is an ε -neighbourhood of the set of singular points of Z , ∂Z_ε its border, and $\text{Area } \partial Z_\varepsilon$ the Riemannian area of the smooth part of ∂Z_ε .

If (2.1) is satisfied, it is easy to see that Z represents a cycle in homology of M . It is well known (see e.g. [GH], Chapter 0, §2), that any closed

complex analytic cycle satisfies (2.1). One defines the Riemannian area $\text{Area}_g(Z)$ of Z as the Riemannian area of the smooth part of Z . If (2.1) holds, the Riemannian area is well defined.

Now, let M be a hyperkähler manifold, and $Z \subset M$ a compact real analytic cycle which satisfies (2.1). Given an induced complex structure L , consider the symplectic area of Z ,

$$V_Z(L) := c \int_Z \omega_L^n,$$

normalized by a constant $c = \frac{1}{n!2^n}$, in such a way that $V_Z(I) = \text{Area}_g(Z)$ for complex analytic subvarieties of (M, I) .

Definition 2.1: In the above assumptions, consider V_Z as a function $V_Z : \mathbb{C}P^1 \longrightarrow \mathbb{R}$ associating to $L \in \mathbb{C}P^1$ the symplectic area of Z . Then V_Z is called **the trisymplectic volume function**. Obviously, V_Z is determined by the homology class of Z .

If the cycle Z is trianalytic, the function $V_Z : \mathbb{C}P^1 \longrightarrow \mathbb{R}$ is clearly constant. Indeed, in this case $V_Z(L) = \text{Area}_g(Z)$ for all $L \in \mathbb{C}P^1$. It turns out that the converse is also true.

Proposition 2.2: ([V1]) Let M be a hyperkähler manifold, I an induced complex structure, and $Z \subset (M, I)$ a compact complex analytic subvariety. Then Z is trianalytic if and only if the function $V_Z : \mathbb{C}P^1 \longrightarrow \mathbb{R}$ is constant.

Proof: The “only if” part is clear, as we indicated above. Assume, conversely, that V_Z is constant. Then the Riemannian area is equal to symplectic area $V_Z(L)$, for all L . For any real analytic cycle Z on a Kähler manifold satisfying (2.1), one has the following inequality of Riemannian and symplectic area

$$V_Z(L) \leq \text{Area}_g(Z), \tag{2.2}$$

called Wirtenger’s inequality. This inequality is reached if and only if Z is a complex analytic subvariety ([Sto], page 7). Since $V_Z(L)$ is constant, $V_Z(L) = \text{Area}_g(Z)$ for all L . This implies that Z is complex analytic in (M, L) , for all L . ■

3 Critical values of V_Z

Let M be a hyperkähler manifold, $[Z] \in H_{2n}(M, \mathbb{Z})$ a cycle in cohomology, and $V_{[Z]} : \mathbb{C}P^1 \rightarrow \mathbb{R}$ the trisymplectic area function defined above. It is easy to see that $V_{[Z]}$ is a polynomial, in the coordinates

$$a, b, c \in \mathbb{R}, a^2 + b^2 + c^2 = 1$$

on $\mathbb{C}P^1 \cong S^2$. Indeed, if $L \in \mathbb{C}P^1$ is a point corresponding to $a, b, c \in \mathbb{R}$, $L = aI + bJ + cK$, then

$$\omega_L = a\omega_I + b\omega_J + c\omega_K,$$

and

$$V_{[Z]}(L) = \frac{1}{n!2^n} \int_{[Z]} (a\omega_I + b\omega_J + c\omega_K)^n.$$

Proposition 3.1: Let M be a hyperkähler manifold, I an induced complex structure, $Z \subset (M, I)$ a compact complex analytic subvariety, and $V_Z : \mathbb{C}P^1 \rightarrow \mathbb{R}$ the corresponding trisymplectic area function. Then V_Z has a maximum in I .

Proof: By Wirtinger's inequality (2.2), $V_Z(L) \leq \text{Area}_g(Z)$ for all induced complex structures $L \in \mathbb{C}P^1$, and $V_Z(I) = \text{Area}_g(Z)$.

Definition 3.2: Let $I \in \mathbb{C}P^1$ be an induced complex structure. Assume that for all integer cycles $[Z] \in H_{2n}(M, \mathbb{Z})$, the function $V_{[Z]} : \mathbb{C}P^1 \rightarrow \mathbb{R}$ is either constant or does not have a maximum in I . Then I is called **generic**.

Claim 3.3: There is at most a countable number of induced complex structures which are not generic.

Proof: As we have mentioned above, $V_{[Z]}$ is a polynomial. There is a countable number of cycles $[Z] \in H_{2n}(M, \mathbb{Z})$, and a finite number of maxima for non-constant $V_{[Z]}$. When we remove all the maxima, for all $[Z] \in H_*(M, \mathbb{Z})$, we obtain the set of generic complex structures. ■

Theorem 3.4: Let M be a hyperkähler manifold, and I a generic induced complex structure. Then all compact complex analytic subvarieties in (M, I) are trianalytic.

Proof: Follows from Proposition 2.2 and Proposition 3.1. ■

Remark 3.5: A trianalytic subvariety is not necessarily smooth. However, its singularities are very simple. In [V4] it was shown that any trianalytic subvariety $Z \subset M$ is an image of a hyperkähler immersion $\tilde{Z} \longrightarrow M$, with \tilde{Z} smooth and hyperkähler.

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